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Motion of a vortex near a free surface†

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The early motion of a single vortex suddenly placed near a free surface is studied analytically. The general initial/boundary-value problem is solved in terms of a Taylor expansion in time. The vortex position and the surface elevation are determined to third order. We find a precise distinction between subcritical (weak) and supercritical (strong) vortices. All vortices start with retrograde horizontal motion. After a short time, subcritical vortices tend to turn and continue their motion in the prograde direction. Supercritical vortices cannot turn, but will continue their retrograde motion. They will accumulate a surface mound until surface breaking eventually occurs.

1. Introduction

Two-dimensional inviscid flow has the distinctive feature that vorticity is conserved for each fluid particle (Helmholtz's theorem). This fundamental relation follows immediately from the curl of Euler's equation of motion. It means that vortex stretching, being a fundamental mechanism in turbulence, cannot exist in strictly two-dimensional flow.

Basic implications of Helmholtz's theorem for vortices near a free surface still remain to be investigated. In the present work we consider the simplest possible case; the vorticity is zero everywhere except for one singular vortex. Helmholtz's theorem is only relevant if the vortex is free; not forced by any exterior agency. A forced vortex is a relevant model for the far field of a moving submerged body with circulation. If the motion of the body is prescribed, the vortex velocity is given *a priori*, completely independent of Helmholtz's theorem. Some basic work has been done by Salvesen & von Kerczek (1976), who studied numerically the free-surface flow due to a single vortex in forced horizontal motion with constant velocity.

We assume the semi-infinite fluid to be undisturbed except for a single vortex, which is suddenly placed in the fluid at time zero. Within the theory of inviscid flow, this can only be achieved if our single vortex is generated impulsively as a starting vortex, shed from a body with circulation prescribed by the Kutta condition. We will neglect all influence from the impulsively started body on the free-surface flow. This is only possible if we consider an extremely slender two-dimensional foil, which starts impulsively to more vertically with constant velocity, from a position at rest close to the free surface. Provided the angle of attack is extremely small and the impulsive velocity very large, the foil leaves no signature on the free-surface flow, except for its shed vortex. Owing to Kelvin's theorem of the conservation of circulation (Batchelor 1967, p. 273), the circulation of the free vortex has the same magnitude (but opposite sign) as the circulation of the submerged foil in its steady vertical motion.

† With an Appendix by R. P. Tong.

The present nonlinear problem has never been studied theoretically.† Just recently, several investigations have been done on a closely related problem: a counter-rotating vortex pair which is generated impulsively and moves vertically with a nonlinear interaction with the free surface. The vortex-pair problem has been investigated experimentally and numerically by Willmarth *et al.* (1989), numerically by Telste (1989) and Marcus & Berger (1989) and analytically by Tyvand (1990*a*). Earlier work in this field is more restricted: either the free-surface condition is linearized (Novikov 1981), or the surface is treated as a rigid lid (Lamb 1932, p. 223; Sarpkaya & Henderson 1984).

The technique of power series expansion in time for free-surface flows has been applied in particular to singular wavemaker problems (Peregrine 1972; Greenhow & Lin 1983). Although the series is asymptotic with zero radius of convergence, it is suited to a basic classification of single free vortices near a free surface: we find a precise distinction between subcritical and supercritical vortices. A similar distinction will not exist for a vortex pair (Tyvand 1990*a*), so the cases of one and two vortices are fundamentally different.

The present analysis also gives an asymptotic expression for the leading gravity wave radiated out from the vortex at small times. Tyvand (1990*b*) applied the present method to a strong source which is turned on below a free surface. It produces a leading wave of the classical Cauchy–Poisson type (Lamb 1932, p. 385), but with a smoother start than in the cases of initial impulse or initial surface hump. The leading gravity wave from a vortex will tend more rapidly to zero in the far field than that from a source.

If viscosity is taken into account, there are several possibilities for generating an isolated vortex with axis parallel to a free surface. An example is the vortex shedding due to the roll motion of a ship. If the Reynolds number of such a vortex is large enough, the present model is relevant for its early time evolution. One basic problem is the importance of wave radiation out from the vortex region. A related question is how large is the surface elevation that can be accumulated before surface breaking occurs? The present study tends to support the conclusion by Telste (1989): a vortex near a free surface will never send out much wave energy. Either the vortex is too weak to be able to deform the surface significantly, or surface breaking occurs quickly.

2. Mathematical formulation

We consider an inviscid semi-infinite fluid initially at rest in the gravity field. A horizontal line vortex with circulation Γ is put impulsively into the fluid at time $t = 0$. Positive circulation is defined in the counter-clockwise direction. The initial depth of submergence below the free surface is denoted by D. The gravitational acceleration is denoted by g.

The fluid is at rest for $t < 0$. The submerged vortex is free and moves with the fluid velocity according to Helmholtz's theorem. A Cartesian coordinate system is defined, with x-axis in the undisturbed free surface and y-axis vertically upwards. We define $x = 0$ by the initial location of the vortex.

We introduce dimensionless quantities by defining D as unit of length, Γ/D as unit

† Note added in proof: Recently this problem has been studied numerically by Yu & Tryggvason (1990).

of velocity, and D^2/Γ as unit of time. We have one characteristic dimensionless number; the Froude number defined by

$$F = \Gamma/(gD^3)^{\frac{1}{2}}, \tag{2.1}$$

The dimensionless velocity potential is denoted by $\Phi(x, y, t)$. The vortex location is given by $x = X(t)$ and $y = -Y(t)$. The dimensionless surface elevation is denoted by $\eta(x, t)$.

The initial/boundary-value problem may be expressed as follows:

$$\nabla^2\Phi = 0 \quad \text{in the fluid except for the vortex point } (\mathrm{X}, -\mathrm{Y}), \tag{2.2}$$

$$|\nabla\Phi| \to 0, \quad y \to -\infty, \tag{2.3}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Phi}{\partial y}, \quad y = \eta(x, t), \tag{2.4}$$

$$\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}|\nabla\Phi|^2 + F^{-2}\eta = 0, \quad y = \eta(x, t), \tag{2.5}$$

$$\Phi(x, 0, 0) = 0, \tag{2.6}$$

$$\eta(x, 0) = 0. \tag{2.7}$$

Condition (2.6) is derived by integrating Bernoulli's equation (2.5) over the extremely short time interval of impulse start. This is in accordance with Wehausen & Laitone (1960, equation 13.54). The above equations (2.3)–(2.7) are identical with the equations solved numerically by Telste (1989).

3. Taylor expansion in time

For a thorough investigation of the behaviour for small values of time, it is useful to introduce the Taylor expansion:

$$(\Phi, \eta, X, Y) = (\Phi_0, \eta_0, X_0, Y_0) + t(\Phi_1, \eta_1, X_1, Y_1) + t^2(\Phi_2, \eta_2, X_2, Y_2) + \dots. \tag{3.1}$$

By this approach we extrapolate the behaviour for later time from the information which is available at $t = 0$. Our results are exact, but the asymptotic series is restricted in validity to a short time interval. In contrast, the numerical results for a long-time simulation of the full nonlinear equations must be filtered to suppress instabilities (Telste 1989). The Taylor expansion in time was suggested for nonlinear free-surface problems by Peregrine (1972). He applied it to the free-surface flow due to an impulsively started wavemaker, see also Greenhow & Lin (1983).

By definition we have

$$X_0 = 0, \quad Y_0 = 1, \quad \eta_0 = 0. \tag{3.2}$$

To find the higher-order vortex coordinates, Helmholtz's theorem must be applied most carefully. We will come back to that in the next section.

We will now define the boundary-value problem to each order in the time expansion. This is done by differentiating the full problem (2.2)–(2.7) as many times as necessary, and putting $t = 0$. However, we must be careful to take into account the total time dependence, which implicitly includes the y-coordinate through the surface elevation. That is, the partial time derivatives alone cannot describe the full time variation. All boundary-value problems will now be defined for the undisturbed fluid domain $y < 0$. The field equations are

$$\nabla^2\Phi_n = 0, \quad y < 0, \quad n = 0, 1, 2, \dots \tag{3.3}$$

except for the point $(x, y) = (X, -Y)$, expanded to order t^n.

The far-field conditions are

$$|\nabla \Phi_n| \to 0, \quad y \to -\infty, \quad n = 0, 1, 2, \dots. \tag{3.4}$$

The first three free-surface conditions are

$$\Phi_0 = 0, \quad y = 0, \tag{3.5}$$

$$\Phi_1 = -\frac{1}{2}\left(\frac{\partial \Phi}{\partial y}\right)^2, \quad y = 0, \tag{3.6}$$

$$\Phi_2 = -\frac{\partial \Phi_0}{\partial y}\left[\frac{\partial \Phi_1}{\partial y} + \tfrac{1}{2}(F^{-2})\right], \quad y = 0. \tag{3.7}$$

The leading gravity term enters the problem to second order. Because of the minus second power of the Froude number, it is clear that the time interval for which the asymptotic series is valid shrinks rapidly with decreasing F. However, we do not put any lower limit on the Froude number. This is because we are always interested in the behaviour during the time interval where the series expansion is valid, no matter how short it is.

The higher-order boundary-value problems also involve an equation defining the surface elevation to each order. The first three are as follows:

$$\eta_1 = \frac{\partial \Phi_0}{\partial y}, \quad y = 0, \tag{3.8}$$

$$2\eta_2 = \frac{\partial \Phi_1}{\partial y}, \quad y = 0, \tag{3.9}$$

$$6\eta_3 = 2\frac{\partial \Phi_2}{\partial y} + 2\frac{\partial \Phi_0}{\partial y}\frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^3 \Phi_0}{\partial y^3}\left(\frac{\partial \Phi_0}{\partial y}\right)^2 - 2\frac{\partial^2 \Phi_0}{\partial x \partial y}\frac{\partial \Phi_1}{\partial x} - 2\left(\frac{\partial^2 \Phi_0}{\partial x\, \partial y}\right)^2 \frac{\partial \Phi_0}{\partial y}, \quad y = 0. \tag{3.10}$$

The fact that the leading gravitational contribution enters the surface elevation to third order shows that our choice of dimensionless quantities provides the proper scaling for the problem.

4. On Helmholtz's theorem and the vortex potential

Helmholtz's theorem says that the singular vortex is convected with the non-singular contribution to the total flow field at the vortex location. We need to clarify its application. Let us therefore summarize (3.3)–(3.7), and include Helmholtz's theorem:

$$\nabla^2 \Phi_n = 0, \quad y < 0 \quad \text{except for the point } (x, y) = (X, -Y), \tag{4.1a}$$

$$|\nabla \Phi_n| \to 0, \quad y \to -\infty, \tag{4.1b}$$

$$\Phi_n(x, 0) = f_n(x), \tag{4.1c}$$

$(n+1)(X_{n+1}, -Y_{n+1})$ = the nth-order contribution to the expansion of

$$\left[\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) (\text{non-singular part of } \Phi)|_{(X,-Y)}\right]. \tag{4.1d}$$

These equations are valid for $n = 0, 1, 2, \dots$. The functions $f_n(x)$ introduced in (4.1c) have already been defined in (3.5)–(3.7), for $n = 0$, 1 and 2.

Although the full problem is nonlinear, the expansion (3.1) gives us a sequence of formally linear equations, with the non-linear terms treated as known inhomogeneities to each order. So it is possible to apply the principle of superposition for Laplace's equation. In this connection it is very important to note that the expansion (3.1) involves a simultaneous extrapolation of the free surface and the vortex location from $t = 0$: so in the full final expansions the free surface is taken as $y = 0$ and the vortex position as $(x, y) = (0, -1)$.

First we choose to separate out the singularity of the vortex from the inhomogeneities at the boundary $y = 0$. By the principle of superposition we then write:

$$\Phi = \psi + \phi, \quad \text{to each order } (n). \tag{4.2}$$

Here ψ is the singular vortex potential which takes into account the full vortex singularity, with homogeneous boundary conditions; ϕ is a regular (non-singular) potential which accounts for all inhomogeneous boundary conditions at $y = 0$.

Let us first write the non-expanded version of the vortex potential:

$$\psi = \frac{1}{2\pi} \arctan \frac{y+Y}{x-X} + \frac{1}{2\pi} \arctan \frac{y-Y}{x-X}. \tag{4.3}$$

We thus satisfy the required boundary condition $\psi = 0$ at $y = 0$ by adding a negative (corotating) image vortex in the point (X, Y) outside the fluid domain. The explicit expansion of (4.3) can only be done successively, one order at a time. Even though the vortex is moving, it produces an instantaneous flow field which is the same as if the vortex had been at rest. Therefore we can state this general expression for the vortex field without knowing the motion of the vortex. But in order to expand this vortex potential in time we need to know the vortex motion.

The vortex potential now enters the problem in two ways. (i) By its spatial derivatives to each order in the free surface conditions (3.5)–(3.10). (ii) The non-singular part of its gradient contributes to the vortex motion according to Helmholtz's theorem. This is given by

$$\text{Non-singular part of } \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right)\bigg|_{(X,-Y)} = \left(\frac{1}{4\pi Y}, 0\right). \tag{4.4}$$

We note that only Y but not X to each order will contribute to the vortex motion through the vortex potential.

By combining (4.1*d*), (4.3) and (4.4) we now arrive at an explicit vector formulation of Helmholtz's theorem to order n:

$$(n+1)(X_{n+1}, -Y_{n+1}) = \left(\frac{\partial\phi_n}{\partial x}, \frac{\partial\phi_n}{\partial y}\right)\bigg|_{(0,-1)} + n\text{th-order term in the expansion of } \left(\frac{1}{4\pi Y}, 0\right). \tag{4.5}$$

This equation is recursive, in that the vortex coordinate Y_n must be known (from Helmholtz's theorem to order $n-1$) before Helmholtz's theorem to the order n can be given quantitatively.

In order to apply (4.5), it is necessary to solve the nth order boundary-value problem for the regular potential $(n = 0, 1, 2, \ldots)$:

$$\nabla^2\phi_n = 0, \quad y < 0, \tag{4.6a}$$

$$|\nabla\phi_n| \to 0, \quad y \to -\infty, \tag{4.6b}$$

$$\phi_n(x, 0) = f_n(x). \tag{4.6c}$$

These boundary-value problems for the regular potential involve the following y-derivative of the vortex potential:

$$\frac{\partial \psi}{\partial y}(x,0) = \frac{1}{\pi}\frac{x-X}{(x-X)^2+Y^2} \tag{4.7}$$

expanded to each order. This expansion must also be done successively.

As the zeroth-order regular potential is zero, we can immediately find the zeroth-order vortex motion from the zeroth-order contribution to (4.1):

$$X_1 = \frac{\partial \psi_0}{\partial x}(0,-1) = \frac{1}{4\pi}, \quad Y_1 = \frac{\partial \psi_0}{\partial y}(0,-1) = 0. \tag{4.8}$$

(Some confusion may arise because the vortex motion is of lower order than the vortex position, the former being the derivative of the latter.) We can also take a step further by combining (3.2) and (4.8) and introduce a two-term expansion for Y into (4.4). As the second term in this expansion is zero, the result will be:

$$\text{Non-singular part of } \nabla\psi_1(0,-1) = 0. \tag{4.9}$$

We already know the zeroth- and first-order contributions to (4.7) by invoking (3.2), (3.8) and (4.8):

$$\frac{\partial \psi_0}{\partial y}(x,0) = \eta_1(x) = \frac{x}{\pi(x^2+1)}, \tag{4.10}$$

$$\frac{\partial \psi_1}{\partial y}(x,0) = \frac{x^2-1}{4\pi^2(x^2+1)^2}, \tag{4.11}$$

Equation (4.11) is valid only for free vortices, following Helmholtz's theorem. A more general version of this equation is the following:

$$\frac{\partial \psi_1}{\partial y}(x,0) = \frac{X_1(x^2-1)-2Y_1 x}{\pi(x^2+1)^2}. \tag{4.12}$$

Equation (4.12) is also valid for a forced vortex, with X_1 and Y_1 given *a priori*. It is worth noting that a forced vertical motion of a vortex will produce an antisymmetric contribution to the second-order surface elevation.

Equation (4.10) gives the total first-order surface elevation. It has extremal values at $|x| = 1$. A simple geometrical argument explains why this is so: a single vortex (as well as its corotating image) induces extremal free-surface velocity at points where the position vector makes an angle of 45° with the vertical direction.

5. On the regular potential and the total flow

All higher orders of the regular potential may now be given analytically by Poisson's integral formula for a half-plane (Morse & Feshbach 1953, p. 371):

$$\phi_n(x,y) = -\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\phi_n(s,0)\,\mathrm{d}s}{(x-s)^2+y^2}, \quad n = 1,2,\ldots. \tag{5.1}$$

This integral is evaluated in each case by residue calculus. We are interested in two properties of these regular potentials: (i) the normal derivative at the boundary, which contributes to the surface elevation; (ii) the gradient at the point $(x,y) = (0, -1)$, which contributes to the vortex motion, by Helmholtz's theorem (4.5).

We first solve the first-order regular potential and find

$$\frac{\partial \phi_1}{\partial y}(x, 0) = \frac{x^4 - 6x^2 + 1}{4\pi^2(x^2+1)^3}. \tag{5.2}$$

This is the same as twice the second-order surface elevation that would have resulted from a static vortex. It arises from the nonlinear convective acceleration, and would have been absent if we had linearized the Bernoulli equation (2.5). Equation (3.9) shows that half the sum of (4.11) and (5.2) defines the total second-order surface elevation:

$$\eta_2(x) = \frac{x^2(x^2-3)}{4\pi^2(x^2+1)^3}. \tag{5.3}$$

The second-order elevation is negative for $|x| < \surd 3$ and positive for $|x| > \surd 3$.

We now calculate the gradient of the first-order potential at the vortex location:

$$\nabla \phi_1(0, -1) = 0. \tag{5.4}$$

Owing to symmetry, it is obvious that the horizontal component is zero, but it is surprising that the vertical component is also zero. By Helmholtz's theorem we add together (4.9) and (5.4) to get

$$X_2 = \frac{1}{2}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \phi_1}{\partial x}\right)\Bigg|_{(0,-1)} = 0, \quad -Y_2 = \frac{1}{2}\left(\frac{\partial \psi_1}{\partial y} + \frac{\partial \phi_1}{\partial y}\right)\Bigg|_{(0,-1)} = 0. \tag{5.5}$$

To second-order the vortex will then move in a straight line with constant velocity, just as to first order.

We will now study the second-order potential and the third-order surface elevation. These are the leading orders to which gravity effects enter the problem, through the Froude number. In the dimensionless initial/boundary-value problem the Froude number is the only parameter which measures the vortex strength relative to depth of submergence and gravity.

Because of the extremely complicated nonlinear terms in (3.10) we will not find the general third-order elevation, but only its gravity-dependent part:

$$\eta_3^{(F)} = -\frac{x}{3\pi F^2(x^2+1)^2}, \tag{5.6}$$

which has extremal values at $|x| = \frac{1}{3}\surd 3 = 0.5774$. Here we note a sign difference compared with the first-order elevation. This has the physical explanation that the radiation of gravity waves reduces the amplitudes of the surface crest and trough around $|x| = 1$. The leading gravity waves are defined as the asymptotic limits of (5.6) as $|x| \to \infty$.

We have calculated the gradient of the second-order regular potential at the vortex location:

$$\frac{\partial \phi_2}{\partial x}(0, -1) = \frac{1}{32\pi^3} - \frac{1}{8\pi F^2}, \quad \frac{\partial \phi_2}{\partial y}(0, -1) = 0. \tag{5.7}$$

It is obvious that the vertical component is zero because of symmetry. Invoking (5.5) we find from Helmholtz's theorem (4.5):

$$3X_3 = \frac{1}{8\pi}\left(\frac{1}{4\pi^2} - \frac{1}{F^2}\right), \quad Y_3 = 0. \tag{5.8}$$

Here the contribution from the vortex potential is zero. Equation (5.8) defines the second-order vortex velocity components.

6. Subcritical versus supercritical vortices

The sign of the third-order vortex position is very important because it determines whether the vortex will go on moving to the right, or turn and go back. This gives an exact definition of critical vortex strength, which can only be achieved analytically. The Froude number F_c for a vortex of critical strength is defined by $X_3 = 0$, which by (5.8) leads to

$$F_c = 2\pi. \tag{6.1}$$

Strictly speaking, the Froude number F, (2.1), must be defined at the initial instant, as the effective depth will change during the motion. The vortex always starts to move into a region with positive surface elevation, and thereby digs itself more deeply into the surrounding fluid. Although the vortex moves strictly horizontally to our calculated order, it will increase its effective depth. It might make sense to say that the effective Froude number is reduced during the early motion, but we will not pursue this further.

As criterion (6.1) is exact, it suggests the definition of an alternative Froude number $\hat{F}$ (all quantities dimensional):

$$\hat{F} = \frac{\Gamma}{2\pi(gD^3)^{\frac{1}{2}}} = \frac{\Gamma/(2\pi D)}{(gD)^{\frac{1}{2}}} = \frac{V}{c}. \tag{6.2}$$

Let us define the concept of the 'touching circle': it is a circle with its centre at the vortex location, with radius D. So the undisturbed free surface is tangent to the touching circle. In (6.2), V denotes the velocity induced by the vortex (not its image) at the perimeter of the touching circle. Alternatively, V is twice the initial velocity of the vortex. Furthermore, c denotes the phase velocity for the 'characteristic water wave' associated with the vortex. This is either a hydrostatic shallow-water wave in a canal with depth D or, alternatively, a linear deep-water wave with wavelength equal to the perimeter of the touching circle. The ratio V/c, as a final version of the definition (6.2), shows the analogy with the uniform free-surface flow in a canal, where the critical Froude number is 1. Our modified Froude number ($\hat{F}$) will also have 1 as its critical value.

Subcritical (weak) vortices can now be defined by

$$\hat{F} < 1. \tag{6.3}$$

According to the present theory, subcritical vortices will move in the retrograde direction (opposite that of water waves with the same direction of particle circulation) during the dimensionless time interval

$$0 < t < 2\pi(2/(\hat{F}^{-2}-1))^{\frac{1}{2}} \tag{6.4}$$

then stop and move backwards, in the prograde direction, which is the ordinary direction of motion for a weak vortex (Lamb 1932, p. 223). However, the present theory will certainly be invalid outside the interval given by (6.4). This interval is of course very small when $\hat{F} \ll 1$, where we have the asymptotic limit:

$$0 < t < \surd 2F. \tag{6.5}$$

(Here the original Froude number F gives the shortest formula). In dimensional terms, this upper limit of (6.5) may be written

$$t_* = \surd 2D/(gD)^{\frac{1}{2}}. \tag{6.6}$$

To this order of approximation, the time it takes for a very weak vortex to turn is equal to the time it takes for a body in free fall to cover the distance D, starting from rest. This is the same as the time it takes for the characteristic water wave to travel a distance equal to $\sqrt{2}$ times the initial depth of submergence. So, even though our theory is then limited to very small times, it gives essential information about the early motion. But we cannot predict the evolution towards Lamb's solution for larger times. We just note that the magnitude of the large-time prograde velocity is the same as that of the initial retrograde velocity.

We can never expect the present theory to make sense for dimensionless times exceeding 1. This means that for Froude numbers (F) between 1 (say) and the critical value 6.28, we cannot rely on our series expansion when it predicts that the vortex will turn and reverse its motion. We still know that the initial retrograde motion has to be retarded, but surface breaking may occur before a reversing of the motion has been reached.

We can study several other interesting physical features of very weak vortices. Let us ask the question: What is the size of a circle in which the particle rotates exactly one turn by the time a very weak vortex has reversed its motion? The circumference of such a circle (around the vortex) is

$$(\sqrt{2F})^{\frac{1}{2}} \quad (\text{assuming } \hat{F} \ll 1)$$

with D as length unit. We also want to know how far a very weak vortex has travelled in the retrograde direction by the time it turns. The answer is

$$X_{\max} = \frac{\sqrt{2}}{3}\hat{F} + O(\hat{F}^3). \tag{6.7}$$

By that time it has produced a dimensionless surface elevation given by

$$\eta_{\max} = \sqrt{2}\hat{F} + O(\hat{F}^2). \tag{6.8}$$

This surface crest, which is three times the maximal retrograde displacement of the weak vortex, is given solely by the first-order elevation. This result may be extended to stronger vortices, leading to the conclusion that the highest surface elevation predicted by this analytical theory will be of order 10^{-1}.

Supercritical (strong) vortices can now be defined by

$$\hat{F} > 1. \tag{6.9}$$

Not much can be said about this case, because strong nonlinear effects will dominate when t exceeds 1. We believe that supercritical vortices will always lead to surface breaking. It is highly improbable that any supercritical vortex is able to turn and move in the prograde direction, because its initial acceleration is in the retrograde direction, as shown by the present theory. Even if it was later retarded, the surface would most probably break before a turn towards prograde motion could be completed.

7. Summary and conclusions

The initial free-surface flow due to a submerged horizontal vortex has been studied analytically by a Taylor expansion in time. The vortex is inserted into the fluid at time zero, and moves with the non-singular part of the fluid velocity according to Helmholtz's theorem.

The vortex position as a function of time is calculated to third order in the series expansion. To this order, the motion is purely horizontal.

We have introduced a modified Froude number $\hat{F}$ which is the ratio between the induced velocity (by the single vortex) at the distance of the undisturbed surface, and the phase velocity of a characteristic water wave. The critical modified Froude number is shown to be equal to 1. All vortices start with retrograde motion. A subcritical vortex tends to turn after a certain time interval, and it will normally continue to move in the prograde direction. But if its Froude number is sufficiently close to the critical value, there is a possibility that surface breaking (inducing turbulence and dissipation of mechanical energy) will take place before the turn is completed, so the vortex will not be able to move in the prograde direction. The results suggest that all supercritical vortices will lead to surface breaking.

Although the modified Froude number $\hat{F}$ is defined *a posteriori*, it is no artificial definition, as it is the ratio between a flow velocity and a wave velocity. Our distinction between subcritical and supercritical vortices is given by $\hat{F}$ smaller than or greater than 1. This criterion for a critical vortex looks deceivingly simple. So we should emphasize that it includes accumulated effects of nonlinear interactions at two levels: (i) The interaction of the zeroth-order velocity with itself at the surface, through the squared-velocity term in Bernoulli's equation. Alternatively, this may be interpreted as the interaction of the first-order surface elevation with itself. (ii) The interaction between the zeroth- and first-order vertical velocities at the free surface. Alternatively, this may be understood as the interaction between the first- and second-order surface elevations.

Appendix. The interaction of a point vortex with a free surface: numerical solution

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Equations (2.2)–(2.7) are solved numerically using the boundary integral method of Dold & Peregrine (1986) in the modified version described by Tanaka *et al.* (1987) for finite water depth. Using the principle of superposition for Laplace's equation, the velocity potential $\Phi(x, y, t)$ is written as

$$\Phi(x, y, t) = \phi_r(x, y, t) + \phi_s(x, y, t),$$

where ϕ_r is a regular potential function associated with the disturbance of the free surface and ϕ_s accounts for the singularity of the vortex at $(X, -Y)$. In order to reduce the extent of the computational domain ϕ_s is chosen as the sum of an infinite series of image points, giving

$$\phi_s(x, y, t) = \frac{1}{2\pi}\arctan\left(\coth\left[\frac{\pi D}{4h}(x-X)\right]\tan\left[\frac{\pi D}{4h}(y+Y)\right]\right)$$
$$-\frac{1}{2\pi}\arctan\left(\coth\left[\frac{\pi D}{4h}(x-X)\right]\tan\left[\frac{\pi D}{4h}\left(y-Y+\frac{2h}{D}\right)\right]\right)$$

in non-dimensional form. Here h is the depth and h/D is chosen so that the effect of finite depth is small in order to provide a comparison with the infinite-depth solution.

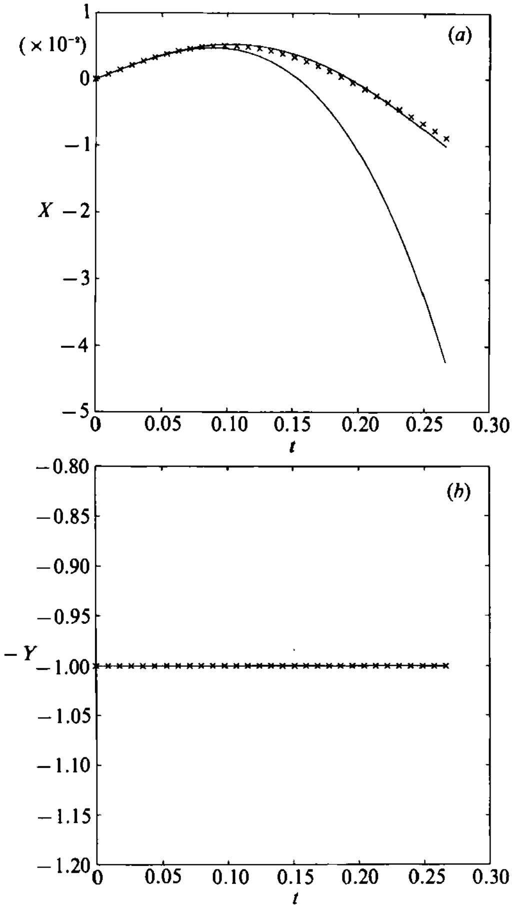

FIGURE 1. Path of vortex at $(X, -Y)$ for $\hat{F} = 0.01$. (*a*) ——, Tyvand's solution $X(t) = 0.07958t - 3.3592t^3$; ×, computed solution. The continuous line close to the computed points shows the effect of adding $6.4t^4$ to Tyvand's solution. (*b*) ——, Tyvand's solution $Y = 1$; ×, computed solution.

The velocity components of ϕ_r are computed by the numerical programme as described in Dold & Peregrine (1986) and Tanaka *et al.* (1987) with the contributions from ϕ_s being added at appropriate points. The motion of the point vortex is defined as

$$\frac{dX}{dt} = \lim_{(x,y)\to(X,\,-Y)} \frac{\partial \Phi}{\partial x}, \quad \frac{d(-Y)}{dt} = \lim_{(x,y)\to(X,\,-Y)} \frac{\partial \Phi}{\partial y}.$$

The computation was carried out for the cases $\hat{F} = 0.01, 0.1, 0.5$ with $h/D = 20$ and the computed path of the point vortex is compared in figures 1–3 with the third-order analytic solution developed by Tyvand in the main body of this paper. For the case $\hat{F} = 0.01$, the numerical solution agrees closely with the analytic one until the vortex changes direction, and then the two solutions diverge. The difference is $O(t^4)$ and it was found that adding an empirically fitted t^4 term to the solution of Tyvand for $X(t)$ gave a good approximation to the numerical solution (figure 1*a*). The largest errors in computing the motion of the point vortex come from two sources: (i) the fact that the integration is carried out over a finite domain; and (ii) the effect of finite depth.

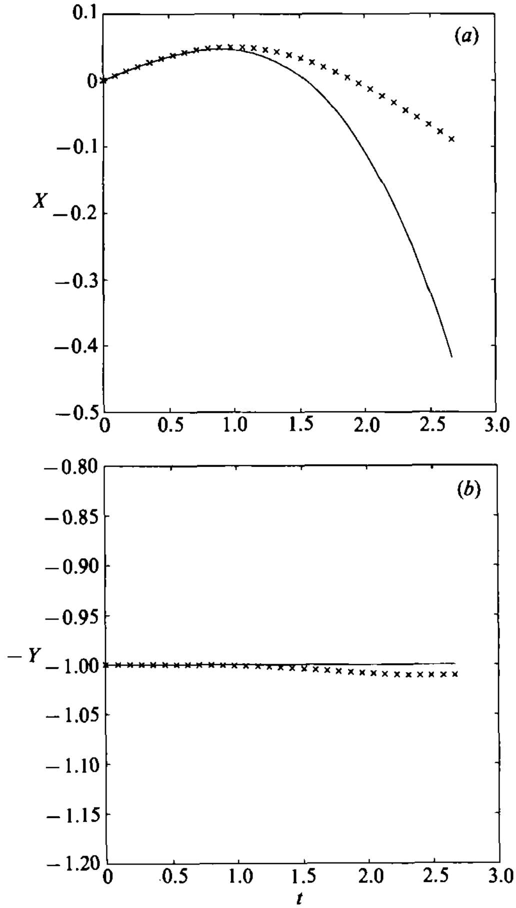

FIGURE 2. Path of vortex at $(X, -Y)$ for $\hat{F} = 0.1$. (a) ——, Tyvand's solution $X(t) = 0.07958t - 0.03326t^3$; ×, computed solution. (b) ——, Tyvand's solution $Y = 1$; ×, computed solution.

These errors are $O(10^{-4})$ and so it appears that, for the time interval under consideration, the difference in the two solutions is largely due to the truncation of the analytic one. The comparison between analytic and numerical solutions shows that the third-order solution gives a good prediction of the time when the vortex changes direction even for the cases $\hat{F} = 0.1$ (figure 2) and $\hat{F} = 0.5$ (figure 3) when the series solution is not strictly valid. When $\hat{F} = 0.5$ the surface steepens appreciably after the vortex changes direction and increases its depth. The computation breaks down as the surface moves towards breaking.

The numerical solution thus confirms the usefulness of the third-order solution produced by Tyvand in describing the initial motion of a point vortex started impulsively beneath a free surface and in predicting the change of direction of a subcritical vortex.

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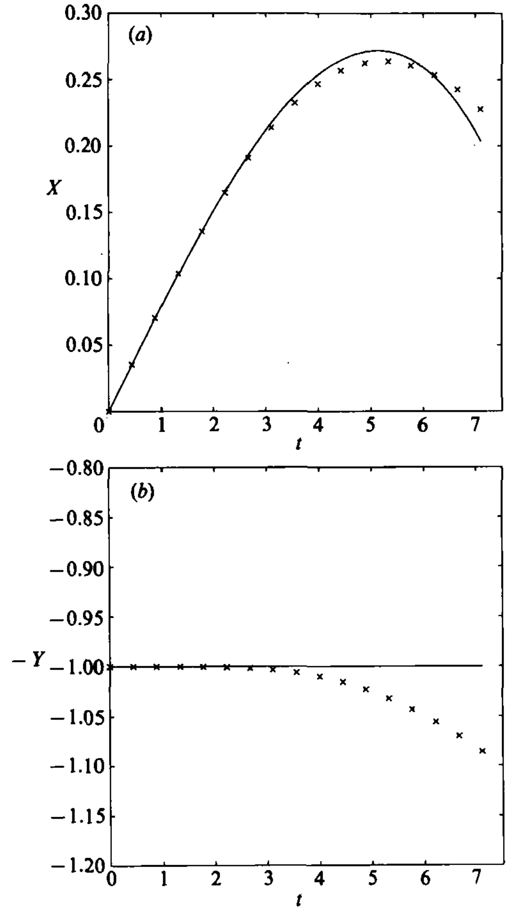

FIGURE 3. Path of vortex at $(X, -Y)$ for $\hat{F} = 0.5$. (*a*) ——, Tyvand's solution $X(t) = 0.07958t - 0.001t^3$; ×, computed solution. (*b*) ——, Tyvand's solution $Y = 1$; ×, computed solution.

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